

## Chapter 16 Appendix

### 16A Real Options Methodology

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In the main body of **Chapter 16** in the printed text we presented the basic principles and general results of option valuation theory as applied to speculative land valuation and basic development project evaluation and decision making. In this online supplement, we will extend **sections 16.4** and **16.5** of the main text to present the details of what you need to know to actually apply the real options model to specific land valuation and simple development project problems. (Applications to more complex development projects are discussed in **Chapter 18**.)

#### 16A.1 The Binomial Model of Option Value

The one-period binomial world in which the development option examples of sections **16.2** and **16.3** were presented is obviously a tremendous simplification of reality. In the real world, time is continuous, and asset values can assume many more than just two possible outcomes. However, the binomial model is more than just a pedagogical simplification device. It is a building block that can be used to construct a much more general and realistic option valuation and analysis tool. This is because the dynamics of the evolution of the



value of built property through time, upon which the land value and the optimal development policy depend, can be modeled as a series of single-period binomial outcome possibilities. We can apply the tools presented in the preceding section within each period. By stitching the individual binomial periods together sequentially, we can span as long a time frame as we like, and by making each individual binomial period as short as we want, we can get the model to approach continuous time and continuous pricing realistically. In this section, we will see how this works in detail.

### 16A1.1 Building the Underlying Asset Value “Tree”

To begin, let’s analyze in a bit more depth how our previous example binomial world was constructed, so that we can see how to extend this model to represent a more realistic and general world. Recall that a new office building today can be observed to be worth \$100 million (if it already exists). Next year, that building (or to be more precise, a similar new one then) will be worth either \$113.21 million with a 70% probability or 78.62 million with a 30% probability. Where did we get these numbers?

In fact, these numbers were selected so that the binomial world we created would converge as the length of time in each period became shorter and the number of sequential binomial periods became greater, toward a continuous time world that has the following characteristics for the dynamics of office building value:

- **9%**      $= r_v$       $=$  *Expected annual total return on investment in completed office buildings.*
- **6%**      $= y_v$       $=$  *Annual net rental income cash payout (yield) as a fraction of current building value.*
- **20%**    $= \sigma_v$     $=$  *Expected annual volatility of returns on individual completed office buildings.<sup>1</sup>*

Note that the above assumptions regarding  $r_v$  and  $y_v$  carry the implication that the expected

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<sup>1</sup> Note that the relevant volatility to use in the context of an option to develop a single asset is the single-asset volatility, which includes the idiosyncratic risk of the individual building and thus is larger than the volatility that would typically be measured for a real estate market index or a portfolio of many properties. (See discussions about real estate volatility in **Chapters 3 and 11**.)



annual growth rate in (new) office building values is:

**Equation A.1**

$$\begin{aligned} g_v &= \frac{1 + r_v}{1 + y_v} - 1 \\ &= \frac{1.09}{1.06} - 1 = 2.83\% \end{aligned}$$

One can start out specifying any two of the three return component variables for the underlying asset,  $r_v$ ,  $y_v$ , or  $g_v$ , and the third will be automatically determined by the relationship in equation (A.1).<sup>2</sup>

We can model the above dynamics in a binomial world as follows. Letting  $V_0$  be the current (observable) value of the underlying asset (the \$100 million in our previous example), the next year's binomial outcome values are set at:

**Equation A.2a**

$$\begin{aligned} V_1^{up} &= V_0 \times \frac{1 + \sigma_v}{1 + y_v} \\ V_1^{down} &= V_0 / \frac{1 + \sigma_v}{1 + y_v} \end{aligned}$$

And the probability of the *up* outcome is set at:

**Equation A.2b**

$$p = ((1 + r_v) - 1/(1 + \sigma_v)) / ((1 + \sigma_v) - 1/(1 + \sigma_v))$$

(The probability of the *down* outcome is, of course, just  $1 - p$ .) Thus, in the case of our example:

$$V_1^{up} = \$100 \times \frac{1.20}{1.06} = \$113.21 \text{ million}$$

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<sup>2</sup>With this in mind, it will often be advisable to make the estimates of  $g_v$  and  $y_v$  as realistic as possible (letting  $r_v$  follow from those estimates) because it turns out (as we shall see later) that the results of the option model are independent of the value chosen for  $r_v$ . In this regard, note that the growth rate measured by  $g_v$  is that in the value of new buildings in the market, not reflecting the depreciation in existing structures. (Recall our discussions in **Chapters 7 and 9** regarding the real depreciation of existing structures.)



$$V_1^{down} = \$100 / \frac{1.20}{1.06} = \$78.62 \text{ million}$$

$$p = \frac{1.09 - 0.83}{1.20 - 0.83} = \frac{0.26}{0.37} = 70\%$$

Now, let's shorten the length of the time periods we are modeling from one year to one month. Supposing that our previously given return and volatility parameters represent nominal annual rates, to shorten the periods to months, we divide the given nominal annual rates by 12 for the return components and by the square root of 12 for the volatility. We still use formulas (A.1) and (A.2) in the model, only now we have for each period (month):  $r_v = 9\%/12 = 0.75\%$ ;  $y_v = 6\%/12 = 0.50\%$ ,  $g_v = (1.0075/1.0050) - 1 = 0.25\%$ , and  $\sigma_v = 20\%/\sqrt{12} = 20\%/3.46 = 5.77\%$ . Thus, applying equation A.1, we have:

$$V_1^{up} = \$100 \times \frac{1.0577}{1.005} = \$105.25 \text{ million}$$

$$V_1^{down} = \$100 / \frac{1.0577}{1.005} = \$94.07 \text{ million}$$

$$p = \frac{1.0075 - 0.9454}{1.0577 - 0.9454} = \frac{0.062}{0.112} = 55.27\%$$

The *up* outcome is now \$105.25 million with a 55.27% probability, and the down outcome is \$94.07 million with a 44.73% probability *next month*.

From each of these two future possible “*states of the world*,” two further future states of the world are possible the month after that. Thus, we can extend our binomial projection to that second month (two months from the present), assuming that the same one-period dynamics apply to the underlying asset in every period. From next month's “*up*” outcome, the possible month 2 outcomes will be:

$$V_{0,2} = \$105.25 \times \frac{1.0577}{1.005} = \$110.77 \text{ million}$$

$$V_{1,2} = \$105.25 / \frac{1.0577}{1.005} = \$99.01 \text{ million}$$

From next month's “*down*” outcome, the two further possibilities in month 2 are:

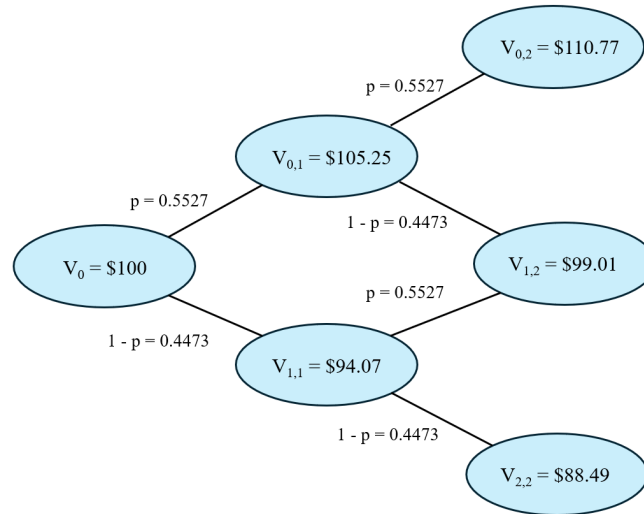
$$V_{1,2} = \$94.07 \times \frac{1.0577}{1.005} = \$99.01 \text{ million}$$

$$V_{2,2} = \$94.07 / \frac{1.0577}{1.005} = \$88.49 \text{ million}$$



(Notice that we are adopting a labeling convention in which the asset is given subscripts “ $i$ ” and “ $j$ ” that characterize the state of the world, with “ $i$ ” representing the total number of *down* outcomes since the present and “ $j$ ” representing the total number of periods since the present.) These results are presented graphically in **Exhibit 16A-1**.

#### EXHIBIT 16A-1 Extending the Binomial Tree to a Second Month



Notice a remarkable (and very important) feature of the above procedure: the *down* outcome from next month’s *up* scenario is *the same value* as the *up* outcome from next month’s *down* scenario, namely, \$99.01 million. This is no accident. Mechanically, it results simply from the commutative property of multiplication and our assumption of the underlying asset’s constant volatility and payout rate. From any current underlying asset<sup>3</sup> value  $V_{ij}$ , the *up* outcome is  $V_{i,j+1} = V_{ij} / (1 + y_v)$  times  $(1 + \sigma_v)$ , and the *down* outcome is  $V_{i+1,j+1} = V_{ij} / (1 + y_v)$  divided by  $(1 + \sigma_v)$ , the multiplicative inverse of each other. Starting from any  $V_{ij}$  value, two periods later, unless there have been two *up* moves in a row or two

<sup>3</sup> Note that an assumption of a constant expected return (or growth rate) in the underlying asset is not technically required in this option valuation model. We can modify the growth or return expectations by modifying the assumed values of  $g_v$  or  $r_v$  in any future state of the world (holding  $y_v$  constant). For example, we could make the expected return (and growth rate) be a function of the time period of the value of the underlying asset or both (so as to represent a cyclical or “mean reverting” market for the underlying asset, for example). Constant growth is, however, the classical assumption (and the simplest baseline case). It corresponds to an asset market that is sufficiently efficient so as to have “memoryless” prices (i.e., a market in which future returns are not predictable based on past returns because current asset prices reflect all relevant currently available information).



*down* moves in a row, the outcome will be either *up times down* or *down times up*, which in either case, because of commutativity and inverse cancellation end up at  $V_{i+1,j+2} = V_{i,j}/(1 + y_v)^2$ .

From a computational perspective, these mechanics are vital, as they cause the future possible values of the underlying asset (the future states of the world) to “recombine” at intermediate points, greatly reducing the number of possible future states of the world that must be considered.<sup>4</sup> More importantly, from an economic perspective, these mechanics cause the model to converge toward a “normal” (Gaussian) distribution for the underlying asset returns as the time periods become shorter (assuming constant volatility).

Note also that while the binomial probability of the single-period *up* and *down* outcomes from either of the two possible month 1 values of the new office building remain 55.27% and 44.73%, respectively, from the standpoint of the present (“month 0”), the probabilities of the *three* possible future values of the underlying asset two months from now are:

$$\begin{aligned} \text{Prob}[V_{0,2} = \$110.77] &= (0.5527)(0.5527) &&= 30.55\% \\ \text{Prob}[V_{1,2} = \$99.01] &= (0.5527)(0.4473) + (0.4473)(0.5527) &&= 49.44\% \\ \text{Prob}[V_{2,2} = \$88.49] &= (0.4473)(0.4473) &&= \underline{20.00\%} + \\ &&&100.00\% \end{aligned}$$

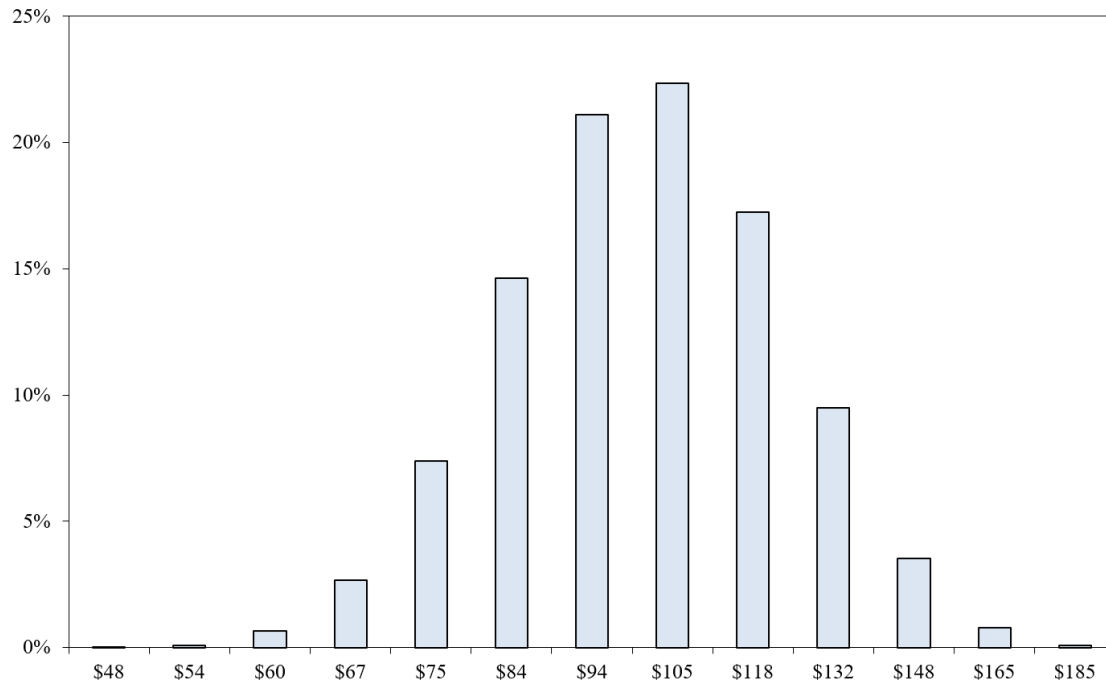
Of course, these three probabilities sum to 100%. The larger middle probability reflects the fact that there are two “routes” by which the value can end up at \$99.01 million in two periods from \$100 million today (either *up* followed by *down* or *down* followed by *up*) while there is only one “route” by which the other two values can be obtained (either two *ups* or two *downs*, respectively).

**Exhibit 16A-2** One Year (Period 12) Value Probabilities for the Underlying Asset (new office building)

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<sup>4</sup> Without this recombination property, the number of future possible states of the world that we would have to consider  $n$  periods in the future would be  $2n$ . With recombination, the number of future states simply equals  $n + 1$ . Thus, to consider 12 monthly periods into the future, with recombination, we have 13 states of the world at the end of the year, and without recombination, we would have 4,096 states of the world.





Suppose we continue building the underlying asset binomial “tree” forward in time out more and more periods in the same manner as we just did, out to a total of 12 periods (1 year) from the present. And suppose we enumerate all of the resulting possible underlying asset value outcomes one year in the future and tabulate the probability, as of the present, of each of those possible outcomes (the “unconditional” probabilities) in the same manner as above (i.e., multiplying the probabilities across steps and counting up the number of “routes” to each outcome). Then, the probability distribution of the future possible values of the office building one year from now will be modeled as shown graphically in Exhibit CD27-2. Notice that this is the type of “bell-shaped” probability distribution that we would realistically expect.<sup>5</sup> The mean of this value distribution is  $E_0[V_{12}] = V_0((1 + r_v)/(1 + y_v))^{12} = V_0(1 + g_v)^{12} = \$100 \times (1.0075/1.005)^{12} = \$100 \times 1.0025^{12} = \$103$  million. Its standard deviation as a fraction of the starting value of \$100 million is 20%, thus reflecting the growth dynamics parameters (mean and volatility) we input into the model. This demonstrates how; even with as few as 12 periods per year, the binomial model can well

<sup>5</sup> To be more precise, the binomial model with constant dynamics, as we have described here, will converge toward a normally distributed (symmetric) continuously compounded return distribution, which equates to a log-normal distribution of possible future asset values (which is slightly skewed to the right, as negative values of the underlying asset are not possible).



represent a realistic model of underlying asset value dynamics in terms of mean, volatility, and general shape of the probability distribution.

### 16A.1.2 Evaluating the Option

**Exhibit 16A-3** shows the complete underlying asset value tree for the 12-month office building development option. (Note that the first two columns reflect the previously calculated numbers.) Each  $i,j$  cell in the table represents a future possible state of the world between now and next year.

**Exhibit 16A-3** Underlying Asset Value Tree for 12 Months

Office building values ("V" ex-dividend, new):													
Month ("j"):	0	1	2	3	4	5	6	7	8	9	10	11	n = 12
"down" moves ("i"):													
0	\$ 100.00	\$ 105.25	\$ 110.77	\$ 116.58	\$ 122.70	\$ 129.14	\$ 135.91	\$ 143.05	\$ 150.55	\$ 158.45	\$ 166.77	\$ 175.52	\$ 184.73
1		\$ 94.07	\$ 99.01	\$ 104.20	\$ 109.67	\$ 115.43	\$ 121.48	\$ 127.86	\$ 134.57	\$ 141.63	\$ 149.06	\$ 156.88	\$ 165.11
2			\$ 88.49	\$ 93.14	\$ 98.02	\$ 103.17	\$ 108.58	\$ 114.28	\$ 120.28	\$ 126.59	\$ 133.23	\$ 140.22	\$ 147.58
3				\$ 83.25	\$ 87.62	\$ 92.21	\$ 97.05	\$ 102.14	\$ 107.50	\$ 113.15	\$ 119.08	\$ 125.33	\$ 131.91
4					\$ 78.31	\$ 82.42	\$ 86.75	\$ 91.30	\$ 96.09	\$ 101.13	\$ 106.44	\$ 112.02	\$ 117.90
5						\$ 73.67	\$ 77.53	\$ 81.60	\$ 85.89	\$ 90.39	\$ 95.13	\$ 100.13	\$ 105.38
6							\$ 69.30	\$ 72.94	\$ 76.77	\$ 80.79	\$ 85.03	\$ 89.49	\$ 94.19
7								\$ 65.19	\$ 68.61	\$ 72.21	\$ 76.00	\$ 79.99	\$ 84.19
8									\$ 61.33	\$ 64.55	\$ 67.93	\$ 71.50	\$ 75.25
9										\$ 57.69	\$ 60.72	\$ 63.91	\$ 67.26
10											\$ 54.27	\$ 57.12	\$ 60.12
11												\$ 51.05	\$ 53.73
12													\$ 48.03

To evaluate the option, the next step is to identify construction cost values corresponding to each future time period, reflecting the expected growth in construction costs. The table in **Exhibit 16A-4** presents such values corresponding to each state of the world represented previously in **16A-3**, reflecting our previous assumption of an expected construction cost growth rate of 2% per year, starting from \$88.24 million today and growing to \$90 million next year.<sup>6</sup> Consistent with our previous assumption of riskless construction costs (recall that construction of the office building was going to cost \$90 million no matter which scenario occurred), all the cost figures are the same within each column (i.e., there is no uncertainty about future construction costs).<sup>7</sup>

<sup>6</sup> The 2% rate is a simple annual rate. The equivalent nominal annual rate with monthly compounding is 1.98%, implying a simple monthly rate of  $(\$90.00/\$88.24)^{1/12} - 1 = 0.1652\%$ .

<sup>7</sup> This simplifying assumption might reflect, for example, the use of fixed-price construction contracts. Risky construction costs can be incorporated into the model by a mathematical transformation, computing the option value *per dollar of construction cost*. This transformation is discussed in **section 16A2** in the context of the Samuelson-McKean formula.



The final step is to compute the option values corresponding to each state of the world. This is done using the same certainty-equivalence DCF technique we described in **section 16.3.2**, applied one cell at a time, starting from the terminal period (in this case, month 12), working backward in time from right to left across the table. (Note that while the underlying value tree is built from left to right, working forward in time, the option value tree based on that underlying value tree is built from right to left, working backward in time.)

**EXHIBIT 16A-4** Construction Cost Tree for 12 Months (in \$ millions)

Office Building Construction Costs ("K"):													
Period ("j"):	0	1	2	3	4	5	6	7	8	9	10	11	12
"down" moves ("i"):													
0	\$ 88.24	\$ 88.38	\$ 88.53	\$ 88.67	\$ 88.82	\$ 88.97	\$ 89.11	\$ 89.26	\$ 89.41	\$ 89.56	\$ 89.70	\$ 89.85	\$ 90.00
1		\$ 88.38	\$ 88.53	\$ 88.67	\$ 88.82	\$ 88.97	\$ 89.11	\$ 89.26	\$ 89.41	\$ 89.56	\$ 89.70	\$ 89.85	\$ 90.00
2			\$ 88.53	\$ 88.67	\$ 88.82	\$ 88.97	\$ 89.11	\$ 89.26	\$ 89.41	\$ 89.56	\$ 89.70	\$ 89.85	\$ 90.00
3				\$ 88.67	\$ 88.82	\$ 88.97	\$ 89.11	\$ 89.26	\$ 89.41	\$ 89.56	\$ 89.70	\$ 89.85	\$ 90.00
4					\$ 88.82	\$ 88.97	\$ 89.11	\$ 89.26	\$ 89.41	\$ 89.56	\$ 89.70	\$ 89.85	\$ 90.00
5						\$ 88.97	\$ 89.11	\$ 89.26	\$ 89.41	\$ 89.56	\$ 89.70	\$ 89.85	\$ 90.00
6							\$ 89.11	\$ 89.26	\$ 89.41	\$ 89.56	\$ 89.70	\$ 89.85	\$ 90.00
7								\$ 89.26	\$ 89.41	\$ 89.56	\$ 89.70	\$ 89.85	\$ 90.00
8									\$ 89.41	\$ 89.56	\$ 89.70	\$ 89.85	\$ 90.00
9										\$ 89.56	\$ 89.70	\$ 89.85	\$ 90.00
10											\$ 89.70	\$ 89.85	\$ 90.00
11												\$ 89.85	\$ 90.00
12													\$ 90.00

Thus, we begin by computing the values the option will have in each of the 13 states of the world that are possible for month 12. For example, in state 0,12 the new office building would be worth \$184.73 million, and in state 1,12 it would be worth \$165.11 million (see column 12 in **Exhibit 16A-3**). In either case, the construction cost is \$90 million. Thus, the NPV of immediate development (recall our instantaneous construction assumption) would be \$184.73 – \$90 = \$94.73 million in state 0,12, or \$165.11 – \$90 = \$75.11 million in state 1,12. As the option expires after period 12, its values in all the other period 12 states can be computed similarly as:

$$C_{i,12} = \text{MAX}[V_{i,12} - 90, 0]$$

In this way, all of column 12 in the option value tree is filled in.

To demonstrate the remainder of the option valuation process, let us begin for illustrative purposes by computing the value of a European option, expiring in one year, to develop the office building. That is, suppose the office building can only be developed in month



12, not before. Consider the value of this option one month prior to its expiration, in the highest possible state of the world at that time, state 0,11. In this state of the world, the only possible future value outcomes for the option are either the 0,12 state value of \$94.73 million or the 1,12 state value of \$75.11 million that we previously computed. The value of the option is therefore computed using the certainty-equivalence formula (1) presented in **section 16.3**:

$$\begin{aligned}
 C_{0,11} &= \frac{E[C_{12}] - (C_{0,12} - C_{1,12}) \frac{E[r_v] - r_f}{V_{0,12}\% - V_{1,12}\%}}{1 + r_f} \\
 &= \frac{((0.5527)\$94.73 + (0.4473)\$75.11 - (\$94.73 - \$75.11) \frac{0.0075 - 0.0025}{1.0577 - \frac{1}{1.0577}})}{1.0025} \\
 &= \frac{\$85.95 - (\$19.26) \frac{0.005}{0.1123}}{1.0025} = \frac{\$85.08}{1.0025} = \$84.87
 \end{aligned}$$

In this way, we compute the option's value in each state of the world in period (column) 11, in each case based on the two possible option values in the following period. Once we have computed the option values in all of the period 11 states, we can move back to the period (column) 10 and compute all the option values in that period in the same manner. We can thus work all the way back to value the option in the present (time 0). Exhibit 16A-5 shows a present value of \$11.09 million for the option to develop in month 12 (but not before).

**EXHIBIT 16A-5** European Option Value Tree for 12-Month Option

Development is possible only in month 12:

Period ("j"): "down" moves ("i"):	0	1	2	3	4	5	6	7	8	9	10	11	12
0	\$ 11.09	\$ 14.67	\$ 19.01	\$ 24.10	\$ 29.89	\$ 36.31	\$ 43.25	\$ 50.64	\$ 58.47	\$ 66.77	\$ 75.56	\$ 84.87	\$ 94.73
1		\$ 7.45	\$ 10.27	\$ 13.84	\$ 18.23	\$ 23.41	\$ 29.32	\$ 35.82	\$ 42.80	\$ 50.19	\$ 58.03	\$ 66.32	\$ 75.11
2			\$ 4.58	\$ 6.63	\$ 9.38	\$ 12.96	\$ 17.43	\$ 22.75	\$ 28.79	\$ 35.38	\$ 42.36	\$ 49.75	\$ 57.58
3				\$ 2.48	\$ 3.81	\$ 5.73	\$ 8.42	\$ 12.02	\$ 16.61	\$ 22.14	\$ 28.35	\$ 34.93	\$ 41.91
4					\$ 1.12	\$ 1.85	\$ 2.99	\$ 4.74	\$ 7.33	\$ 10.99	\$ 15.83	\$ 21.69	\$ 27.90
5						\$ 0.38	\$ 0.67	\$ 1.19	\$ 2.09	\$ 3.59	\$ 6.04	\$ 9.85	\$ 15.38
6							\$ 0.07	\$ 0.14	\$ 0.28	\$ 0.55	\$ 1.08	\$ 2.12	\$ 4.19
7								\$ -	\$ -	\$ -	\$ -	\$ -	\$ -
8									\$ -	\$ -	\$ -	\$ -	\$ -
9										\$ -	\$ -	\$ -	\$ -
10											\$ -	\$ -	\$ -
11												\$ -	\$ -
12													\$ -



The valuation of the American option (that allows development in *any* month prior to the expiration of the option) proceeds in the same way, only at each node, the value of the option is the *maximum* of either its immediate exercise value or the present value of holding the option one more period. The latter value is computed similarly to the European option, using the certainty-equivalence DCF formula. Thus, the complete formula for the valuation of the American option in the binomial world comes in two parts, embedded in an algorithm. The first step in the algorithm is to compute the option's value in the period of its expiration, which is the same as for the European option, as given in (A.3a) below. Supposing option expiration in period  $j = T$ :

**Equation A.3a**

$$C_{i,T} = \text{MAX}[V_{i,T} - K, 0]$$

Next, the option value is computed in each state of the world  $i, j$  prior to the terminal period working backward in time (first for all  $i$  in  $j = T - 1$ , then for all  $i$  in  $j = T - 2$ , and so on, back finally to  $j = 0$ ). The option valuation formula in each state  $i, j$  for  $j < T$  (where  $i$  is the number of *down* outcomes since the present and  $j$  is the total number of periods since the present) is given in equation (A.3b) below.

**Equation A.3b**

$$C_{i,j} = \text{MAX} \left[ V_{i,j} - K_j, \frac{E[C_{j+1}] - (C_{i,j+1} \$ - C_{i+1,j+1} \$) \frac{E[r_v] - r_f}{V_{i,j+1} \% - V_{i+1,j+1} \%}}{1 + r_f} \right]$$

$$= \text{MAX} \left[ V_{i,j} - K_j, \frac{(pC_{i,j+1} + (1-p)C_{i+1,j+1}) - (C_{i,j+1} \$ - C_{i+1,j+1} \$) \frac{E[r_v] - r_f}{(1 + \sigma_v) - 1/(1 + \sigma_v)}}{1 + r_f} \right]$$

where  $p$  is as defined in equation (A.2b).

In our numerical example, the present value of the option is \$12.57 million, as seen in



## Exhibit 16A-6.<sup>8</sup>

### EXHIBIT 16A-6 American Option Value Tree for 12-Month Option

		Development possible in any month:																								
Period ("j"):		0	1	2	3	4	5	6	7	8	9	10	11	12												
"down" moves ("i"):																										
0	\$	12.57	\$	16.87	\$	22.24	\$	27.91	\$	33.88	\$	40.17	\$	46.80	\$	53.79	\$	61.14	\$	68.90	\$	77.06	\$	85.67	\$	94.73
1			\$	8.19	\$	11.41	\$	15.58	\$	20.85	\$	26.46	\$	32.37	\$	38.60	\$	45.16	\$	52.07	\$	59.35	\$	67.03	\$	75.11
2				\$	4.91	\$	7.15	\$	10.22	\$	14.28	\$	19.47	\$	25.02	\$	30.87	\$	37.03	\$	43.53	\$	50.37	\$	57.58	
3					\$	2.61	\$	4.02	\$	6.08	\$	8.99	\$	12.94	\$	18.10	\$	23.59	\$	29.38	\$	35.48	\$	41.91		
4						\$	1.16	\$	1.91	\$	3.11	\$	4.95	\$	7.68	\$	11.57	\$	16.73	\$	22.17	\$	27.90			
5							\$	0.38	\$	0.69	\$	1.22	\$	2.14	\$	3.70	\$	6.25	\$	10.28	\$	15.38				
6								\$	0.07	\$	0.14	\$	0.28	\$	0.55	\$	1.08	\$	2.12	\$	4.19					
7									\$	-	\$	-	\$	-	\$	-	\$	-	\$	-	\$	-				
8										\$	-	\$	-	\$	-	\$	-	\$	-	\$	-					
9											\$	-	\$	-	\$	-	\$	-	\$	-						
10												\$	-	\$	-	\$	-	\$	-							
11													\$	-	\$	-	\$	-								
12														\$	-	\$	-									

Note also that with an American option, it is possible to identify states of the world in which immediate exercise of the option is optimal and those in which it is optimal to continue holding the option. In the real options model of land, these would correspond to states of the world in which immediate development is triggered and those in which the land is continued to be held for speculation. **Exhibit 16A-7** shows the optimal exercise binomial tree for our example 12-month option. Note that it is optimal at present to continue holding the land for speculation and that the first state in which development would be optimal would occur two months from now, if and only if there are two *up* outcomes in the value of office buildings between now and then (state 0,2). In that case, the value of a new office building would at that time be observed to have grown to \$110.77 million from today's \$100 million value.

Note that in the binomial optimal exercise tree, there is, in general, a *boundary line* above and to the right of which it is optimal to exercise and below and to the left of which it is optimal to continue holding the option. This boundary line can be defined by a ratio of current  $V_{ij}/K_j$  values that must be sufficiently high in order to trigger optimal immediate

<sup>8</sup> Note that the modeled option value tends to "oscillate" as a function of whether the option's lifetime is modeled using an even or odd number of periods. For example, if we modeled our example one-year option using 11 equal-length periods (each one 12/11 months long), we would get a valuation of \$12.51 million instead of \$12.57 million. However, if we modeled the one-year option using 10 periods (each 1/10th of a year long), then we would get an option valuation of \$12.58 million. As we previously saw in Section 12.3, with only one period (of one year length), the option was valued at \$12.09 million. The exact relationship between the period length and the convergence to the continuous time valuation is complex and depends upon the parameters of the problem.



development. In other words, the construction benefit-cost ratio must be high enough to make it worthwhile to give up the option premium, the potential value of waiting. In general, this “*hurdle ratio*” will be higher the greater the volatility ( $\sigma_v$ ) and the lower the cash payout rate ( $y_v$ ) in the underlying asset, the lower the growth rate in the exercise price ( $g_k$ ), and the longer the time until expiration of the option. As option expiration approaches, the value of the option premium diminishes toward zero, and exercise becomes optimal as the current value of the underlying asset barely exceeds its current construction cost.

**EXHIBIT 16A-7** American Option Optimal Exercise for 12-Month Option

Period ("j"): "down" moves ("i"):	0	1	2	3	4	5	6	7	8	9	10	11	n = 12
0	hold	hold	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
1		hold	hold	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
2			hold	hold	hold	exer	exer	exer	exer	exer	exer	exer	exer
3				hold	hold	hold	exer	exer	exer	exer	exer	exer	exer
4					hold	hold	hold	exer	exer	exer	exer	exer	exer
5						hold	hold	hold	exer	exer	exer	exer	exer
6							hold	hold	hold	hold	exer	exer	exer
7								hold	hold	hold	hold	hold	exer
8									hold	hold	hold	hold	hold
9										hold	hold	hold	hold
10											hold	hold	hold
11												hold	hold
12													hold

### 16A1.3 Accounting for Construction Time

In all of the option valuation examples we have considered so far, we have assumed that construction was instantaneous, resulting in the immediate completion of a fully operating building as soon as the development decision is made. This, of course, is unrealistic, and to make our development option model realistic, we need to account for the time it takes to complete construction. This is often referred to as the “**time to build**.” Under the assumption that the construction time is known in advance (or is a deterministic function of the time period or of the construction cost, or any variable that is determined in the binomial tree), it is very straightforward to account for time to build-in the option valuation model.

The effect of time to build is that when the decision to exercise the development option is made (the decision to begin construction), instead of immediately obtaining a completed, fully operating building, one obtains a *forward claim* on a fully operating building. A “forward claim” is an enforceable contract that will give its holder the claimed asset at a



definite future point in time. For example, suppose the option is exercised in month  $t$ , and the time to build is 18 months. Then, upon exercise at time  $t$  we obtain, for certain, a fully operating building in month  $t + 18$ . Thus, the *value at time  $t$*  of what one obtains immediately at time  $t$  upon exercise at time  $t$  is the *present value* as of time  $t$  of the forward claim, that is, the present value as of time  $t$  of a fully operating building at time  $t + 18$ . What is the value of this forward claim?

The present value of a forward claim on an asset is a straightforward DCF problem, equal to the expected value (as of the present) of the future value of the asset (at the time of the forward claim), discounted to present at the risk-adjusted discount rate that reflects the asset's opportunity cost of capital.<sup>9</sup> Thus, for example, in the case of our previously considered office building that is currently worth \$100 million (if it were existing completed new at time 0), the expected value of a similar newly completed building 18 months from now is:

$$\begin{aligned} E_0[V_{18}] &= V_0(1 + g_v)^{18} = V_0 \left( \frac{(1 + r_v)}{(1 + y_v)} \right)^{18} \\ &= \$100(1.0025)^{18} = \$100 \left( \frac{(1.0075)}{(1.0050)} \right)^{18} = \$104.57 \text{ million} \end{aligned}$$

And the present value (as of time 0) of the forward claim on this asset is this expectation discounted at the 9%/year (or 0.75% per month) OCC of office buildings:<sup>10</sup>

$$\begin{aligned} PV_0[V_{18}] &= \frac{E_0[V_{18}]}{(1 + r_v)^{18}} = \frac{V_0(1 + g_v)^{18}}{(1 + r_v)^{18}} = \frac{V_0 \left( \frac{(1 + r_v)}{(1 + y_v)} \right)^{18}}{(1 + r_v)^{18}} = \frac{V_0}{(1 + y_v)^{18}} \\ &= \frac{\$104.57}{(1.0075)^{18}} = \frac{\$100}{(1.0050)^{18}} = \$91.41 \text{ million} \end{aligned}$$

<sup>9</sup> Note that a forward claim is not an option; it does not provide the flexibility to forego the claim in an unfavorable outcome. A forward claim is an irrevocable commitment to obtain the asset at a future point in time. Hence, the expected return on an investment in the asset is the OCC appropriate as the discount rate to determine the present value of the claim.

<sup>10</sup> It is somewhat intuitive that the present value of a forward claim on an asset equals its current value discounted by its cash yield rate ( $y_v$ ), as it is only the cash yield component of the asset's total return that is given up in the interim while the investor does not yet possess the asset. The investor will obtain all of the capital value growth in the asset when she takes possession of the asset as a result of the forward claim.



The same is true regarding the construction cost. We have been quoting construction cost as a single lump-sum payment reflecting the entire cost of (instantaneous) construction as of the time when the building is newly complete, and we have assumed a growth trend in these costs of 2% per year (simple annual rate, or 0.1652%/month) in our example. Assuming a convention of quoting the lump-sum construction cost as of the time when the building is newly complete, the forward construction cost 18 months from now that will be due upon completion of our building, consistent with the similarly defined time 0 construction cost of \$88.24 million that we quoted previously (for a building newly complete at time 0), would be  $\$88.24 \times (1.001652)^{18} = \$90.90$  million. The present value (as of time 0) of such a forward construction cost, assuming the OCC for construction cost is the risk-free interest rate of 3% per year (or  $3\%/12 = 0.25\%$  per month), would be:  $\$90.90/(1.0025)^{18} = \$86.90$  million. Thus, as of time 0, the relevant underlying asset value for our previously described development option with 18 months to build is \$91.41 million instead of \$100 million, and the relevant exercise price is \$86.90 million instead of \$88.24 million.

In general, the valuation of the option with time to build is exactly the same as the valuation with instantaneous exercise that we described in **section 16.1.2**, except that we replace the current actual values of the underlying asset  $V_{i,j}$  and of the construction cost  $K_j$  with new values that represent the current values of the forward claims on the underlying asset and the construction cost. Labeling the current (period  $j$ ) values of these forward claims as  $v_{i,j}$  and  $k_j$ , respectively, in the option valuation procedure, we simply replace  $V_{i,j}$  and  $K_j$  with:

$$v_{i,j} = \frac{V_{i,j}}{(1 + y_v)^{ttb}}$$

$$k_j = K_j \left( \frac{(1 + g_k)}{(1 + r_f)} \right)^{ttb} = K_j \left( \left( \frac{(1 + r_f)}{(1 + y_k)} \right) / (1 + r_f) \right)^{ttb} = \frac{K_j}{(1 + y_k)^{ttb}}$$

respectively (where  $ttb$  is the time to build measured in number of periods of the binomial



model). This replacement is done both in the value trees and in the valuation formulas, and then we proceed as before.

Thus, for example, the option we described previously only now with an 18-month construction time would be worth \$7.48 million, as seen in **Exhibit 16A-8**, instead of the \$12.57 million we calculated before with instantaneous construction. It makes sense that the development option would be worth less when construction takes 18 months than when construction is instantaneous, as the time to build prevents the developer from obtaining 18 months worth of net rental cash flows that otherwise could be obtained with instantaneous construction.

The general effect of construction time is to reduce the option value and to increase the currently observable price of similar new buildings, which triggers optimal immediate development. This latter effect causes the likelihood of immediate development to fall (and the expected time until development to increase) as a function of the time to build. For example, suppose the ratio  $(v_{i,j}/k_j)^*$  is just sufficient to trigger optimal immediate development with time to build; then this corresponds to an observable current value trigger ratio of:

$$\left(\frac{V_{i,j}}{K_j}\right)^* = \left(\frac{1+y_v}{1+y_h}\right)^{ttb} \left(\frac{v_{i,j}}{k_j}\right)^* = \left(\frac{(1+r_v)/(1+g_v)}{(1+r_f)/(1+g_k)}\right)^{ttb} \left(\frac{v_{i,j}}{k_j}\right)^*$$

As  $y_v$  will normally be greater than  $y_k$ , the trigger ratio measured in terms of the currently observable values of the underlying asset and construction cost  $(V_{i,j}/K_j)^*$  will be an increasing function of the time to build ( $ttb$ ).<sup>11</sup>

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<sup>11</sup> Normally,  $r_v$  will be greater than  $r_f$ , as the underlying asset is risky, while  $g_v$  and  $g_k$  will probably be about the same magnitude (approximately equal to the general inflation rate). Thus,  $y_v$  will be greater than  $y_k$ .



## EXHIBIT 16A-8 12-Month Binomial Value Trees with 18-Month Time to Build

Underlying Asset Value Tree with Time to Build:

PV of forward claim on new office building (ex-dividend)

Period ("j")	0	1	2	3	4	5	6	7	8	9	10	11	n = 12
"down" moves ("i"):													
0	\$ 91.41	\$ 96.21	\$ 101.26	\$ 106.57	\$ 112.16	\$ 118.05	\$ 124.24	\$ 130.76	\$ 137.63	\$ 144.85	\$ 152.45	\$ 160.45	\$ 168.87
1		\$ 85.99	\$ 90.51	\$ 95.26	\$ 100.25	\$ 105.51	\$ 111.05	\$ 116.88	\$ 123.01	\$ 129.47	\$ 136.26	\$ 143.41	\$ 150.93
2			\$ 80.90	\$ 85.14	\$ 89.61	\$ 94.31	\$ 99.26	\$ 104.47	\$ 109.95	\$ 115.72	\$ 121.79	\$ 128.18	\$ 134.91
3				\$ 76.10	\$ 80.09	\$ 84.30	\$ 88.72	\$ 93.37	\$ 98.27	\$ 103.43	\$ 108.86	\$ 114.57	\$ 120.58
4					\$ 71.59	\$ 75.34	\$ 79.30	\$ 83.46	\$ 87.84	\$ 92.45	\$ 97.30	\$ 102.40	\$ 107.78
5						\$ 67.34	\$ 70.88	\$ 74.60	\$ 78.51	\$ 82.63	\$ 86.97	\$ 91.53	\$ 96.33
6							\$ 63.35	\$ 66.68	\$ 70.17	\$ 73.86	\$ 77.73	\$ 81.81	\$ 86.10
7								\$ 59.60	\$ 62.72	\$ 66.01	\$ 69.48	\$ 73.12	\$ 76.96
8									\$ 56.06	\$ 59.00	\$ 62.10	\$ 65.36	\$ 68.79
9										\$ 52.74	\$ 55.51	\$ 58.42	\$ 61.48
10											\$ 49.61	\$ 52.21	\$ 54.95
11												\$ 46.67	\$ 49.12
12													\$ 43.90

Construction cost with Time to Build:

PV of 18-mo forward cost of construction

Period ("j")	0	1	2	3	4	5	6	7	8	9	10	11	n = 12
"down" moves ("i"):													
0	\$ 86.90	\$ 87.04	\$ 87.19	\$ 87.33	\$ 87.48	\$ 87.62	\$ 87.77	\$ 87.91	\$ 88.06	\$ 88.20	\$ 88.35	\$ 88.49	\$ 88.64
1		\$ 87.04	\$ 87.19	\$ 87.33	\$ 87.48	\$ 87.62	\$ 87.77	\$ 87.91	\$ 88.06	\$ 88.20	\$ 88.35	\$ 88.49	\$ 88.64
2			\$ 87.19	\$ 87.33	\$ 87.48	\$ 87.62	\$ 87.77	\$ 87.91	\$ 88.06	\$ 88.20	\$ 88.35	\$ 88.49	\$ 88.64
3				\$ 87.33	\$ 87.48	\$ 87.62	\$ 87.77	\$ 87.91	\$ 88.06	\$ 88.20	\$ 88.35	\$ 88.49	\$ 88.64
4					\$ 87.48	\$ 87.62	\$ 87.77	\$ 87.91	\$ 88.06	\$ 88.20	\$ 88.35	\$ 88.49	\$ 88.64
5						\$ 87.62	\$ 87.77	\$ 87.91	\$ 88.06	\$ 88.20	\$ 88.35	\$ 88.49	\$ 88.64
6							\$ 87.77	\$ 87.91	\$ 88.06	\$ 88.20	\$ 88.35	\$ 88.49	\$ 88.64
7								\$ 87.91	\$ 88.06	\$ 88.20	\$ 88.35	\$ 88.49	\$ 88.64
8									\$ 88.06	\$ 88.20	\$ 88.35	\$ 88.49	\$ 88.64
9										\$ 88.20	\$ 88.35	\$ 88.49	\$ 88.64
10											\$ 88.35	\$ 88.49	\$ 88.64
11												\$ 88.49	\$ 88.64
12													\$ 88.64

American Option Value Tree with Time to Build:

Period ("j")	0	1	2	3	4	5	6	7	8	9	10	11	n = 12
"down" no Development possible in any month, takes 18 months from then:													
0	\$ 7.48	\$ 10.45	\$ 14.33	\$ 19.24	\$ 24.69	\$ 30.43	\$ 36.48	\$ 42.85	\$ 49.57	\$ 56.65	\$ 64.10	\$ 71.95	\$ 80.23
1		\$ 4.45	\$ 6.51	\$ 9.32	\$ 13.07	\$ 17.89	\$ 23.29	\$ 28.97	\$ 34.96	\$ 41.26	\$ 47.91	\$ 54.92	\$ 62.30
2			\$ 2.35	\$ 3.63	\$ 5.50	\$ 8.15	\$ 11.78	\$ 16.56	\$ 21.89	\$ 27.52	\$ 33.44	\$ 39.69	\$ 46.27
3				\$ 1.04	\$ 1.72	\$ 2.79	\$ 4.44	\$ 6.90	\$ 10.43	\$ 15.23	\$ 20.51	\$ 26.08	\$ 31.94
4					\$ 0.35	\$ 0.62	\$ 1.10	\$ 1.92	\$ 3.29	\$ 5.52	\$ 8.97	\$ 13.91	\$ 19.14
5						\$ 0.07	\$ 0.13	\$ 0.26	\$ 0.51	\$ 1.00	\$ 1.98	\$ 3.90	\$ 7.69
6							\$ -	\$ -	\$ -	\$ -	\$ -	\$ -	\$ -
7								\$ -	\$ -	\$ -	\$ -	\$ -	\$ -
8									\$ -	\$ -	\$ -	\$ -	\$ -
9										\$ -	\$ -	\$ -	\$ -
10											\$ -	\$ -	\$ -
11												\$ -	\$ -
12													\$ -

### 16A1.4 Risk-Neutral Valuation and the Role of $r_f$

The option valuation procedure we have described here uses the certainty-equivalence approach to DCF valuation presented in **Chapter 16**. This valuation procedure is applied to a future probability distribution of possible values of the underlying asset and the construction costs (and of the derivative option) in which these probabilities and values represent the actual probabilities and actual possible values, the *true dynamics* of the relevant underlying asset market.



An alternative methodology is more traditional in much academic literature and is often presented in general finance courses on option valuation theory. This alternative is what is called “**risk-neutral valuation**” and is based on “**risk-neutral dynamics**.” While the risk-neutral approach is completely equivalent to the approach presented here and gives identical option value solutions, it works with probabilities and future values of the underlying asset that are not “true” but rather what would be consistent within a world in which all investors were “risk-neutral” (that is, they did not care about the risk in their investments).

As investors are obviously not risk-neutral in the real world, the risk-neutral valuation approach may be viewed as simply a mathematical device. However, the nature of this “device” and how it was derived reminds us of something important about the economics that underlies our option valuation procedure (or any economically rigorous option valuation procedure), and in so doing, provides us with a useful insight that can simplify our use of the model. In particular, the equivalence of risk-neutral and certainty-equivalence valuation of the option implies that the input value we assume for the OCC of the underlying asset,  $r_v$ , does not affect the value of the option that we obtain from the model.

Mechanically, the risk-neutral valuation procedure can be defined exactly as we have defined the option valuation procedure here in **section 16A1**, only substituting the risk-free interest rate  $r_f$  for the underlying asset OCC rate  $r_v$  everywhere the latter occurs, that is, both in the computation of the binomial outcome probability  $p$  in equation (A.2b) and in the single-period value discounting equation (1) of **section 16.3.2** of the main text [which was first introduced in Appendix 10C and appears here in **section 16A1** in equation (A.3b)]. Note that in this substitution in (A.3b) the  $E[r_v] - r_f$  term in the numerator becomes  $r_f - r_f$ , and hence cancels out. This eliminates the “risk discount” component of the certainty-equivalence value in the numerator of (A.3b), leaving the certainty equivalence as just the  $E[C_t]$  expectation (which is discounted at the risk-free rate, consistent with a risk-neutral world). However, the modification of the definition of the probability  $p$  based on  $r_f$  instead of  $r_v$  in equation (A.2b) causes  $E[C_t]$  now not to be a true expectation (that is,



not an expectation in the real world), but rather an expectation in a risk-neutral world. (In such a world, the expected total return on the underlying asset going forward would be simply the risk-free rate.)

The reason why the risk-neutral approach and the certainty-equivalent approach are identical and why the value of the underlying asset OCC does not impact the valuation of the option is based on the fundamental economics of how the option valuation procedure works, as described in **section 16.3.3** of the main text. The risk-neutral valuation procedure derives directly, mathematically, from the arbitrage valuation approach described in **section 16.3.1**, which we saw to be equivalent to certainty-equivalence valuation in **section 16.3.2** because both are based on the same fundamental cross-market equilibrium condition, as described in **section 16.3.3**. Thus, *given the underlying asset's value, the option's value* is robust to our assumption of the opportunity cost of capital (discount rate) for the underlying asset. In effect, given the lock-step relationship between a derivative asset and its underlying asset, everything that we need to know about the underlying asset for purposes of evaluating the derivative is incorporated in the underlying asset's current market value,  $V_0$ , given the existence of equilibrium across the relevant asset markets. This is convenient, because it makes our evaluation of the option robust to our (possibly erroneous) assumption about what is the OCC of the underlying asset.

## 16A2 A Perpetual Model in Continuous Time

**Section 16.5** in the main text presents the Samuelson-McKean model of a perpetual American option as a basic tool to develop intuition regarding land value and optimal development timing. However, the formula presented in the main text is a bit simplified (for example, it assumes a constant, riskless construction cost), and it does not fully reflect the extent of the Samuelson-McKean model's ability to address the land development option in a realistic way. In this supplemental chapter, we will explore the model in more depth, showing how to increase the realism of the model's assumptions by allowing for construction costs that grow over time, and by allowing for both time and risk in the construction process. We will also present a detailed numerical example of the application of the Samuelson-McKean model.



### 16A2.1 Accounting for Time to Build and Risky Construction Costs

The classical Samuelson-McKean formula, as presented in **section 16.5** of the main text, assumes that the option's exercise price is riskless and that the option exercise is instantaneous. For the real estate development application, it is more realistic to model the exercise price (construction cost) as possibly having some risk and to consider that construction (the exercise of the option) takes time. The latter fact is often referred to as “time to build.” This section will explain how the classical model can be easily adapted to consider these possibilities.

To allow for uncertainty in construction costs, the Samuelson-McKean formula can be subjected to a transformation. In this transformation, the option is evaluated *per dollar of construction cost*. In other words, construction cost is treated as the “numeraire,” and  $K$  always equals unity. The underlying asset,  $V$ , becomes the benefit/cost ratio of building value divided by construction cost. Apart from that, the model is applied as described in the main text, except that the volatility employed in the model is no longer the volatility simply of the underlying asset but rather that of a portfolio that is long in the underlying asset and short in construction costs. Labeling this volatility  $\sigma_p$ , we have:

#### Equation A4

$$\sigma_p = \sqrt{\sigma_v^2 + \sigma_k^2 - 2\rho_{vk}\sigma_v\sigma_k}$$

Where  $\sigma_k$  is the volatility of construction costs, and  $\rho_{vk}$  is the correlation coefficient between building value and construction costs.

Accounting for construction time, or “time to build,” is a straightforward modification of the basic Samuelson-McKean formula. We simply replace  $V_0$  and  $K_0$  in the formula with the *present values* of the expected *future values* of the new building and its construction cost, respectively, as of the time of completion of the construction project, after its required time to build. Thus, the main text's equation (5a) for the option elasticity remains unaffected, but letting  $t_{tb}$  equal the amount of time required for construction, the main text's equation (5b) for the option value becomes:



**Equation 5b\***

$$C_0 = \left( V^* - K_0 / \exp(y_k(ttb)) \right) \left( \frac{V_0 / \exp(y_v(ttb))}{V^*} \right)^\eta$$

and equation (5c) for the hurdle ratio becomes:

**Equation 5c\***

$$V^* = \frac{K_0 / \exp(y_k(ttb))^\eta}{\eta - 1}$$

It is important to note that the hurdle value of the underlying asset,  $V^*$ , in this context equals the hurdle *present value* of the expected *future value* that the underlying asset will have after the time to build. Therefore, expressing the hurdle benefit/cost ratio in terms of currently observable values of the underlying asset and the construction cost (as of time 0), we have:

**Equation A5**

$$\left( \frac{V_0}{K_0} \right)^* = \exp((y_v - y_k)(ttb)) \frac{\eta}{\eta - 1}$$

In other words, assuming that  $y_v > y_k$  (as is likely), the greater the time required for construction (larger value of  $ttb$ ), the greater will be the hurdle-benefit cost ratio that triggers immediate optimal development, as measured by the current (time 0 observable) values of the underlying asset and its construction cost.

**16A2.2 Numerical Example Application of the Samuelson-McKean Formula**

Let's return to the numerical example in **section 16.3** of the main text as an example of how we would apply the Samuelson-McKean formula. There, we had a land parcel that could be developed today to obtain a new office building that would be worth \$100 million today, with a current construction and development cost exclusive of land totaling \$88.24 million. The development option was valid for only one year; there were only two possible future outcome possibilities next year (either the new building would be worth \$113.21 million or \$78.62 million, with probabilities of 70% and 30%, respectively, and with



construction cost by then of \$90 million in any case), and construction was assumed to be instantaneous. In this situation, the option (the land) was seen to be worth \$12.09 million, and the optimal development timing decision was to wait and hold the land for speculation for possible development next year. The Samuelson-McKean model will allow us to make the development option perpetual, with continuous time and continuous pricing. We can also allow for a realistic time to build and risky construction costs. The underlying assumption is that returns are normally distributed and uncorrelated across time.

Suppose the annual cash yield provided by office buildings is 6%, the construction yield rate is 1% (that is, the expected growth rate in construction cost is 1% per year less than the OCC of construction costs), and the relevant volatility is 20% per year.<sup>12</sup> Further, building the office building will take 18 months (1.5 years). Then, we can apply the Samuelson-McKean formula as follows.

First, the option elasticity is seen to be 3.64, based on equation (5a):

$$\begin{aligned}\eta &= \frac{\left\{ y_v - y_k + \frac{\sigma_v^2}{2} + \left[ \left( y_v - y_k + \frac{\sigma_v^2}{2} \right)^2 + 2y_k\sigma_v^2 \right]^{\frac{1}{2}} \right\}}{\sigma_v^2} \\ &= \frac{\left\{ 0.06 - 0.01 + \frac{(0.20)^2}{2} + \left[ \left( 0.06 - 0.01 + \frac{(0.20)^2}{2} \right)^2 + 2(0.01)(0.06)^2 \right]^{\frac{1}{2}} \right\}}{(0.20)^2} \\ &= 3.64\end{aligned}$$

This means that an investment in the land will have 3.64 times the return volatility and 3.64 times the investment risk of an investment in a completed office building.

Supposing (as in section 16.3) that the OCC of the completed office building is 9% and the

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<sup>12</sup> Apart from the noted differences, these inputs are roughly consistent with the example in the main text **section 16.3**. For example, the 6% cash yield rate could correspond to an expected new building value growth rate of 3% per year and an OCC of 9% on completed building investments ( $r_v = g_v + y_v = 3\% + 6\% = 9\%$ ). However, note that for use in the Samuelson-McKean formula, the inputs are instantaneous rates and, hence, not exactly comparable to the simple annual rates used in **section 16.3**. Also, note that the 20% volatility,  $\sigma_p$ , and the construction cost OCC (the construction cost discount rate,  $r_k$ ) may reflect some risk in construction costs and some correlation between construction costs and building value.



risk-free interest rate is 3% (implying a risk premium of 6% for the office building), then the expected return risk premium in the land investment should be 3.64 times 6%, or 21.82%. This implies that the OCC for the land, equal to the risk-free rate plus this risk premium, is 24.82% = 3% + 21.82%. Interestingly, this is similar to our original ad hoc assumption of 20% for the OCC of the land back in **section 16.2**.<sup>13</sup>

Next, we can compute the hurdle benefit/cost ratio based on formula (A.8) above:

$$\begin{aligned}\left(\frac{V_0}{K_0}\right)^* &= \exp((y_v - y_k)(ttb)) \frac{\eta}{\eta - 1} \\ &= \exp((0.06 - 0.01)(1.5)) \frac{3.64}{3.64 - 1} \\ &= \exp(0.075)1.38 = 1.4866\end{aligned}$$

This means that as soon as the current value of new office buildings like the one we could build equals 1.4866 times the current construction cost, it will be optimal to immediately commit to construction (which will produce the completed building 18 months later). This implies that, at the time of optimal immediate development, the land value will be 24.02% of the expected future value of the completed building at the time of completion of construction.<sup>14</sup>

Thus, as the current construction cost is \$88.24 million, a current value of  $1.4866 \times \$88.24 = \$131.17$  million for the office building would be necessary to trigger immediate optimal development. As the current actual value of the office building we could develop is only \$100 million, it is optimal to wait and hold the land undeveloped for now.

Equivalently, we could compute the present value of the future completed building (after the required construction time of 1.5 years) that would make immediate development optimal. This value is given by formula (5c\*) as \$119.88 million:

<sup>13</sup> However, keep in mind that, unlike in **section 16.2**, all of the rates in this section are instantaneous rates. For example, the instantaneous OCC rate of 9% implies a simple annual total return expectation of  $\exp(0.09) - 1 = 9.42\%$ . The option instantaneous OCC of 24.82% equates to a simple annual rate of  $\exp(0.2482) - 1 = 28.18\%$ .

<sup>14</sup> This is based on the present value of the land as a fraction of the future value of the completed building, assuming that development is just optimal now at time 0.



$$\begin{aligned}
 V^* &= \frac{K_0/\exp(y_k(ttb))\eta}{\eta - 1} \\
 &= \frac{\$88.24/\exp(0.01(1.5))3.64}{3.64 - 1} \\
 &= \$119.88
 \end{aligned}$$

As the actual present value of a forward claim on an office building in 1.5 years is only \$91.39 million [computed as  $(\$100)\exp[(0.09 - 0.06)1.5]/\exp[(0.09)1.5] = \$100/\exp[(0.06)1.5] = \$100/1.0942 = \$91.39$ ], we do not currently have sufficient value in the underlying asset to justify immediate development.

Finally, we can compute the current value of the land based on equation (5b\*) as follows:

$$\begin{aligned}
 C_0 &= \left( V^* - K_0/\exp(y_k(ttb)) \right) \left( \frac{\left( \frac{V_0}{\exp(y_v(ttb))} \right)}{V^*} \right)^\eta \\
 &= \left( \$119.88 - \frac{\$88.25}{1.0151} \right) \left( \frac{\$91.39}{\$119.88} \right)^{3.64} \\
 &= \$12.28
 \end{aligned}$$

Thus, the land is currently worth \$12.28 million.

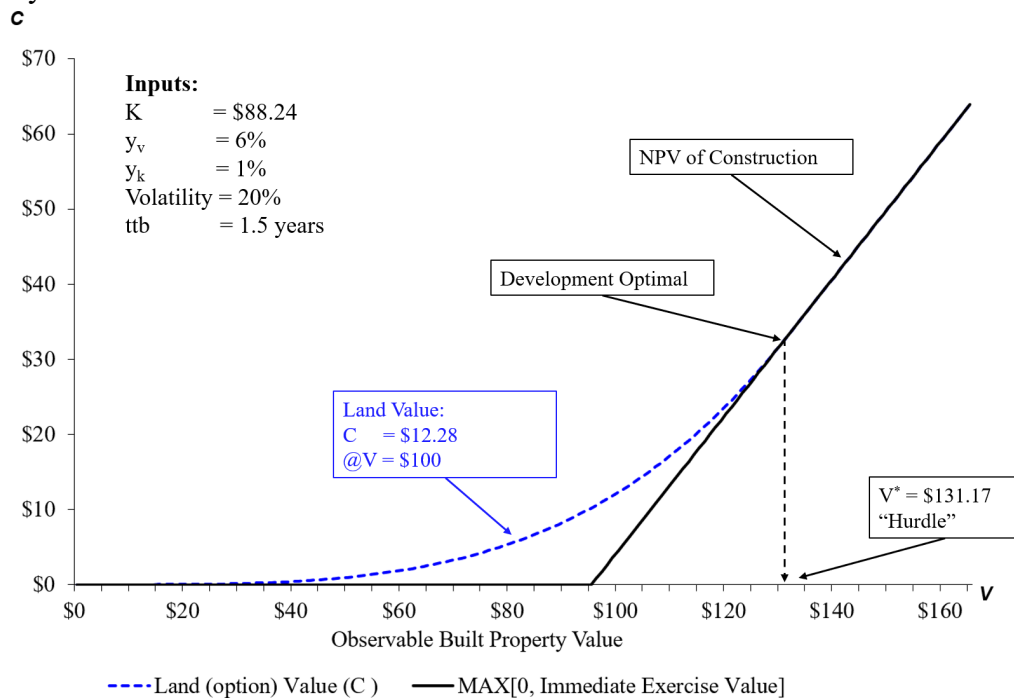
Interestingly, this value is similar to the \$12.09 million that we computed with our very simple one-year binomial example back in **section 16.3**. This similarity is due to two offsetting effects. On the one hand, the perpetual option is worth considerably more than the one-year option, holding everything else the same. In particular, if we could have instantaneous construction, the Samuelson-McKean valuation of the land would be \$16.38 million (instead of \$12.28 million, and the current hurdle value of the office building would be \$121.69 million instead of \$131.17 million). On the other hand, recognizing the 1.5-year time required for construction reduces the value of the option. If we had applied a 1.5-year time to build within the framework of the one-period binomial model of Section 27.3, we would have computed a value of only \$7.83 million (instead of \$12.09 million).

**Exhibit 16-9** displays graphically the numerical example we have just presented. The



horizontal axis represents the currently observable value of the underlying asset,  $V_0$ , and the vertical axis represents the present value of the land (or the development project). The straight line coming out of the horizontal axis at  $V_0 = \$95.11$  million gives the current NPV of a commitment to proceed with development (excluding consideration of the land cost). This value is simply the present value of the forward claim on the new building minus the present value of its construction cost:  $V_0/\exp[y_v(ttb)] - K_0/\exp[y_k(ttb)]$ , which equals zero at  $V_0 = \$95.11$  when  $K_0 = \$88.24$ ,  $y_v = 6\%$ ,  $y_k = 1\%$ , and  $ttb = 1.5$  years. Notice that the Samuelson-McKean land value, given by the upward-sloping convex curved line, exceeds this construction NPV until the point where on the horizontal axis  $V_0$  equals the hurdle value of  $\$131.17$  million, at which point the land (option) is worth  $\$32.96$  million, exactly equal to the construction NPV:  $V_0/\exp[y_v(ttb)] - K_0/\exp[y_k(ttb)] = \$131.17/\exp[0.06(1.5)] - \$88.24/\exp[0.01(1.5)] = \$119.88 - \$86.92 = \$32.96$  million. At current building values of less than  $\$131.17$  million, the *complete* NPV of the development project, including subtracting the opportunity cost of the land (the option given up), would be negative.

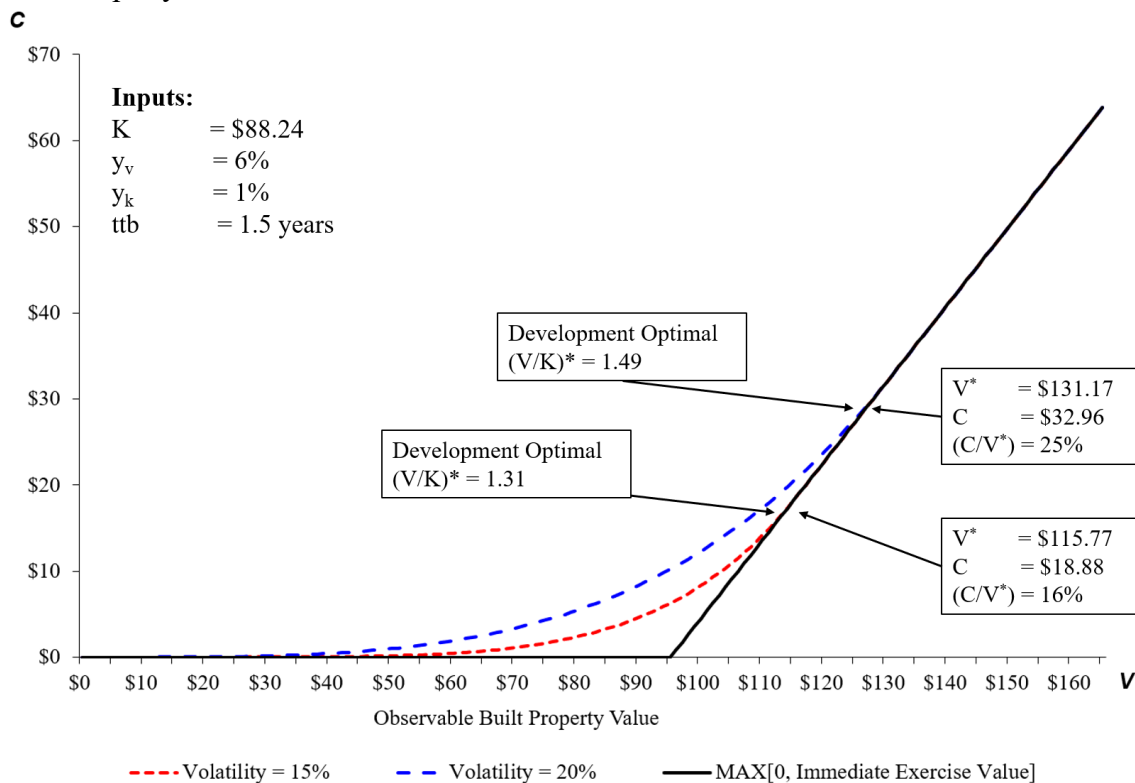
**EXHIBIT 16A-9** Samuelson-McKean Model Land Value as a Function of Current Built Property Value





**Exhibit 16A-10** shows how the option value is affected by a reduction in the relevant volatility. (A similar effect results from an increase in the built property yield rate,  $y_v$ , or from a decrease in the construction yield rate  $y_k$ .) Note that the option value is reduced for all values of the built property, and the hurdle value at which development is optimal is also reduced. Reducing relevant volatility from 20% to 15% reduces the hurdle benefit/cost ratio from 1.49 to 1.31. It reduces the land value fraction of total property value at the moment of optimal development from 25% to 16% of the current property value.<sup>15</sup>

**EXHIBIT 16A-10** Samuelson-McKean Model Land Value as a Function of Current Built Property Value



<sup>15</sup>  $V^*$  in **Exhibit 16A-10** is measured in terms of the currently observable built property value rather than as in equation (5c\*) in terms of the present value of the future building that could be obtained from the development. For example, with 20% volatility, at the point of optimal immediate development, the land value of \$32.96 million equals: (i) 25.1% of the \$131.17 million current observable built property value [ $V_0$ ]; or (ii) 24.0% of the \$137.21 million projected future building value on completion {which equals  $V_0 \exp(g_v(ttb)) = 131.17 \exp[0.03(1.5)]$ }; or (iii) 27.5% of the \$119.88 million present value of the projected completed building value {which equals  $E_0[V_{tb}]/\exp[r_v(ttb)] = \$137.205/\exp[0.09(1.5)]$ }. The land value fraction at the point of optimal development defined in this last way (as a fraction of the present value of the future completed building) simply equals  $1/\eta$ , the inverse of the option elasticity [as  $1/3.64 = 27.5\%$ ], the same as it would be if  $ttb=0$ .